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In the cooperative cost-cumulant control regime for the class of multi-person single-objective decision problems characterized by quadratic random costs and state-feedback information structures, individual decision makers share state information with their neighbors and then autonomously determine decision strategies to achieve the desired goal of the group which is a minimization of a finite linear combination of the first k cost cumulants of a finite-horizon integral quadratic cost associated with a linear stochastic system. Since this problem formation is parameterized by the number of cost cumulants, the scalar coefficients in the linear combination and the group of decision makers, it may be viewed both as a generalization of linear-quadratic Gaussian control, when the first cost cumulant is minimized by a single decision maker and of the problem class of linear-quadratic identical-goal stochastic games when the first cost cumulant is minimized by multiple decision makers. Using a more direct dynamic programming approach to the resultant cost-cumulant initial-cost problem, it is shown that the decision laws associated with multiple persons are linear and are found as the unique solutions of the set of coupled differential matrix Riccati equations, whose solvability guarantees the existence of the closed-loop feedback decision laws for the corresponding multi-person single-objective decision problem.

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Cooperative Solutions in Multi-Person Quadratic Decision Problems: Finite-Horizon and State-Feedback Cost-Cumulant Control Paradigm

Khanh D. Pham
Space Vehicles Directorate
Air Force Research Laboratory
Kirtland AFB, NM 87117 U.S.A.

Abstract— **In the cooperative cost-cumulant control regime for the class of multi-person single-objective decision problems characterized by quadratic random costs and state-feedback information structures, individual decision makers share state information with their neighbors and then autonomously determine decision strategies to achieve the desired goal of the group which is a minimization of a finite linear combination of the first k cost cumulants of a finite-horizon integral quadratic cost associated with a linear stochastic system. Since this problem formulation is parameterized by the number of cost cumulants, the scalar coefficients in the linear combination and the group of decision makers, it may be viewed both as a generalization of linear-quadratic Gaussian control, when the first cost cumulant is minimized by a single decision maker and of the problem class of linear-quadratic identical-goal stochastic games when the first cost cumulant is minimized by multiple decision makers. Using a more direct dynamic programming approach to the resultant cost-cumulant initial-cost problem, it is shown that the decision laws associated with multiple persons are linear and are found as the unique solutions of the set of coupled differential matrix Riccati equations, whose solvability guarantees the existence of the closed-loop feedback decision laws for the corresponding multi-person single-objective decision problem.**

I. Introduction

Cooperative control involves the control of a group of entities that are working collectively and efficiently to solve a problem or meet a common objective. This is an emerging area of research with widespread applications to problems in several engineering disciplines and economic analysis. Now, within the context of performance analysis of cooperative systems, decision laws of cooperative decision makers are adjusted repeatedly until a desired response is reached. It is not at all clear how each decision laws of these decision makers affect the global closed-loop response of a total system. There have been a number of attempts to evaluate the performance of a stochastic system using the average and variance of the associated performance measure. In case the performance measure is normally distributed, these two stochastic moments are sufficient for a full characterization of the probability distribution. However, this is not always the case. It turns out that in order to generalize the results for second-order statistics to higher-order statistics, it is better to consider higher-order cumulants, not higher-order moments. One of main results of the paper shows that the higher-order cumulants of finite-horizon integral quadratic form cost can be obtained directly from the cumulant-generation equation. The result relies heavily on the space-space representation of the class of linear-quadratic decision problems. The other important result consists of optimal decision laws associated with a group of decision makers which simultaneously affect the performance of the cooperative system via a complete statistical description. The paper is structured as follows. The next section prepares the necessary background in generating higher-order statistics which are then used to formulate the cost-cumulant control problem for multiple decision makers. A precise mathematical formulation involving problem statements of the multi-person single-objective decision problem is given next. Finally, the optimal decision laws are presented in the last section with concluding remarks.

Correspondence to Air Force Research Laboratory, AFRL/VSSV, 3550 Aberdeen Ave. SE, Kirtland AFB, NM 87117-5776 U.S.A.; Phone: (505)846-4823; Fax: (505)846-7877; Email: khanh.pham@kirtland.af.mil

II. Performance-Measure Statistics

Let's consider a stochastic decision problem with N cooperative decision makers, identified as $u_1, \ldots, u_N$. Suppose $(t_0, x_0) \in [t_0, t_f] \times \mathbb{R}^n$ is fixed. An input noise $w(t) \triangleq w(t,\omega) : [t_0, t_f] \times \Omega \mapsto \mathbb{R}^p$ is an p-dimensional stationary Wiener process defined with $\{\mathcal{F}_t\}_{t \geq 0}$ being its filtration on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathcal{P})$ over $[t_0, t_f]$ with the correlation of increments

$$E\left\{[w(\tau) - w(\xi)][w(\tau) - w(\xi)]^T\right\} = W|\tau - \xi|, \quad W > 0.$$

Furthermore, decision sets $\mathcal{U}_i \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_i}))$ and $i = 1, \ldots, N$ are assumed to be the subsets of Hilbert space of $\mathbb{R}^{m_i}$-valued square integrable processes on $[t_0, t_f]$ that are adapted to the σ-field $\mathcal{F}_t$ generated by $w(t)$, respectively. Associated with each $(u_1, \ldots, u_N) \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_N$ is a common finite-horizon integral quadratic form (IQF) payoff functional $J : [t_0, t_f] \times \mathbb{R}^n \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_N \mapsto \mathbb{R}^+$ such that

$$J(t_0, x_0; u_1, \ldots, u_N) = x^T(t_f) Q_f x(t_f) + \int_{t_0}^{t_f} \left[x^T(\tau) Q(\tau) x(\tau) + \sum_{i=1}^{N} u_i^T(\tau) R_i(\tau) u_i(\tau) \right] d\tau \quad (1)$$

where the states of the decision problem, $x(t) = x(t,\omega) : [t_0, t_f] \times \Omega \mapsto \mathbb{R}^n$ belong to the Hilbert space $L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^n))$ with $E\left\{\int_{t_0}^{t_f} x^T(\tau) x(\tau) d\tau\right\} < \infty$ and evolve according to the stochastic differential

$$dx(t) = \left[A(t)x(t) + \sum_{i=1}^{N} B_i(t) u_i(t) \right] dt + G(t) dw(t)\,, \quad x(t_0) = x_0 \quad (2)$$

in which $A \in \mathcal{C}([t_0,t_f];\mathbb{R}^{n\times n})$, $B_i \in \mathcal{C}([t_0,t_f];\mathbb{R}^{n\times m_i})$, and $G \in \mathcal{C}([t_0,t_f];\mathbb{R}^{n\times p})$ are deterministic matrix-valued functions together with (A, B_i) uniformly stabilizable. The terminal $Q_f \in \mathbb{R}^{n\times n}$, the state $Q \in \mathcal{C}([t_0,t_f];\mathbb{R}^{n\times n})$, and the control $R_i \in \mathcal{C}([t_0,t_f];\mathbb{R}^{m_i\times m_i})$ weightings are deterministic and positive semidefinite with $R_i(t)$ invertible.

In view of the linear system (2) and the quadratic performance-measure (1), it is reasonable to assume that cooperative decision makers choose their decision laws from a class of memoryless perfect-state strategies, as functions of both time and states, $\gamma_i : [t_0,t_f]\times L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^n)) \mapsto L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^{m_i}))$

$$u_i(t) = \gamma_i(t,x(t)) \triangleq K_i(t)x(t)\,, \tag{3}$$

where the admissible gains $K_i \in \mathcal{C}([t_0,t_f];\mathbb{R}^{m_i\times n})$ are defined in appropriate senses. For a given initial condition $(t_0,x_0) \in [t_0,t_f]\times\mathbb{R}^n$ and subject to these strategies (3), the dynamics of the cooperative decision problem (2) is then given by

$$dx(t) = \left[A(t) + \sum_{i=1}^{N} B_i(t)K_i(t)\right]x(t)dt + G(t)dw(t)\,,$$
$$x(t_0) = x_0\,, \tag{4}$$

and its IQF cost also follows

$$J(t_0,x_0;K_1,\ldots,K_N) = x^T(t_f)Q_f x(t_f) + \int_{t_0}^{t_f} x^T(\tau)\left[Q(\tau) + \sum_{i=1}^{N} K_i^T(\tau)R_i(\tau)K_i(\tau)\right]x(\tau)d\tau. \tag{5}$$

It is now necessary to develop a procedure for generating cost cumulants for the multi-person single-objective decision problem by adapting the parametric method in [3] to characterize a moment-generating function. These cost cumulants are then used to form the performance index in the cost-cumulant control optimization. This approach begins with a replacement of the initial condition (t_0,x_0) by any arbitrary pair (α,x_α). Thus, for the given admissible feedback gains $K_1,\ldots,K_N$, the cost functional (5) is seen as the "cost-to-go", $J(\alpha,x_\alpha)$. The moment-generating function of the vector-valued random process (4) is given by

$$\varphi(\alpha,x_\alpha;\theta) \triangleq \{\exp(\theta J(\alpha,x_\alpha))\}\,, \tag{6}$$

where the scalar $\theta \in \mathbb{R}^+$ is a small parameter. Thus, the cumulant-generating function immediately follows

$$\psi(\alpha,x_\alpha;\theta) \triangleq \ln\{\varphi(\alpha,x_\alpha;\theta)\}\,, \tag{7}$$

in which $\ln\{\cdot\}$ denotes the natural logarithmic transformation of an enclosed entity.

Theorem 1: Cost-Cumulant Generating Function.
For all $\alpha \in [t_0,t_f]$ and the small parameter $\theta \in \mathbb{R}^+$, define

$$\varphi(\alpha,x_\alpha;\theta) \triangleq \varrho(\alpha,\theta)\exp\left(x_\alpha^T\Upsilon(\alpha,\theta)x_\alpha\right)\,, \tag{8}$$
$$\upsilon(\alpha,\theta) \triangleq \ln\{\varrho(\alpha,\theta)\}\,. \tag{9}$$

Then the cost-cumulant generating function is expressed as

$$\psi(\alpha,x_\alpha;\theta) = x_\alpha^T\Upsilon(\alpha,\theta)x_\alpha + \upsilon(\alpha,\theta)\,, \tag{10}$$

where the scalar solution $\upsilon(\alpha,\theta)$ with $\upsilon(t_f,\theta) = 0$ solves

$$\frac{d}{d\alpha}\upsilon(\alpha,\theta) = -\mathrm{Tr}\left\{\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\right\}\,, \tag{11}$$

and the matrix-valued solution $\Upsilon(\alpha,\theta)$ with $\Upsilon(t_f,\theta) = \theta Q_f$ satisfies

$$\begin{aligned}\frac{d}{d\alpha}\Upsilon(\alpha,\theta) = &-\left[A(\alpha) + \sum_{i=1}^{N} B_i(\alpha)K_i(\alpha)\right]^T \Upsilon(\alpha,\theta)\\ &-\Upsilon(\alpha,\theta)\left[A(\alpha) + \sum_{i=1}^{N} B_i(\alpha)K_i(\alpha)\right]\\ &-2\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha,\theta)\\ &-\theta\left[Q(\alpha) + \sum_{i=1}^{N} K_i^T(\alpha)R_i(\alpha)K_i(\alpha)\right].\end{aligned} \tag{12}$$

In addition, the auxiliary solution $\varrho(\alpha,\theta)$ is satisfying the backward-in-time differential equation with $\varrho(t_f,\theta) = 1$

$$\frac{d}{d\alpha}\varrho(\alpha,\theta) = -\varrho(\alpha,\theta)\mathrm{Tr}\{\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\}\,. \tag{13}$$

Proof. For any θ given, let $\varpi(\alpha,x_\alpha;\theta) \triangleq \exp(\theta J(\alpha,x_\alpha))$ then the moment-generating function becomes $\varphi(\alpha,x_\alpha;\theta) = E\{\varpi(\alpha,x_\alpha;\theta)\}$ with the time derivative of

$$\frac{d}{d\alpha}\varphi(\alpha,x_\alpha;\theta) = -\varphi(\alpha,x_\alpha;\theta)\,\theta x_\alpha^T\left[Q(\alpha) + \sum_{i=1}^{N} K_i^T(\alpha)R_i(\alpha)K_i(\alpha)\right]x_\alpha\,.$$

Using the standard Ito's formula, one gets

$$\begin{aligned}d\varphi(\alpha,x_\alpha;\theta) &= E\{d\varpi(\alpha,x_\alpha;\theta)\}\,,\\ &= E\Big\{\varpi_\alpha(\alpha,x_\alpha;\theta)\,d\alpha + \varpi_{x_\alpha}(\alpha,x_\alpha;\theta)\,dx_\alpha\\ &\quad + \frac{1}{2}\mathrm{Tr}\left\{\varpi_{x_\alpha x_\alpha}(\alpha,x_\alpha;\theta)G(\alpha)WG^T(\alpha)\right\}d\alpha\Big\}\,,\\ &= \varphi_\alpha(\alpha,x_\alpha;\theta)\,d\alpha\\ &\quad + \varphi_{x_\alpha}(\alpha,x_\alpha;\theta)\left[A(\alpha) + \sum_{i=1}^{N} B_i(\alpha)K_i(\alpha)\right]x_\alpha d\alpha\\ &\quad + \frac{1}{2}\mathrm{Tr}\left\{\varphi_{x_\alpha x_\alpha}(\alpha,x_\alpha;\theta)\,G(\alpha)\,WG^T(\alpha)\right\}d\alpha\,,\end{aligned}$$

which with the definition (8) leads to

$$\begin{aligned}&-\varphi(\alpha,x_\alpha;\theta)\,\theta x_\alpha^T\left[Q(\alpha) + \sum_{i=1}^{N} K_i^T(\alpha)R_i(\alpha)K_i(\alpha)\right]x_\alpha\\ &= \frac{\frac{d}{d\alpha}\varrho(\alpha,\theta)}{\varrho(\alpha,\theta)}\varphi(\alpha,x_a;\theta) + \varphi(\alpha,x_\alpha;\theta)\,x_\alpha^T\frac{d}{d\alpha}\Upsilon(\alpha,\theta)x_\alpha\\ &+ \varphi(\alpha,x_\alpha;\theta)\left\{x_\alpha^T\left[A(\alpha) + \sum_{i=1}^{N} B_i(\alpha)K_i(\alpha)\right]^T\Upsilon(\alpha,\theta)x_\alpha\right.\\ &\quad\left. + x_\alpha^T\Upsilon_a(\alpha,\theta)\left[A(\alpha) + \sum_{i=1}^{N} B_i(\alpha)K_i(\alpha)\right]x_\alpha\right\}\\ &+ \varphi(\alpha,x_\alpha;\theta)\left\{2x_\alpha^T\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha,\theta)x_\alpha\right.\\ &\quad\left. + \mathrm{Tr}\left\{\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\right\}\right\}.\end{aligned}$$

To have constant and quadratic terms being independent of x_α, it requires that

$$\begin{aligned}\frac{d}{d\alpha}\Upsilon(\alpha,\theta) &= -\left[A(\alpha)+\sum_{i=1}^{N}B_i(\alpha)K_i(\alpha)\right]^T\Upsilon(\alpha,\theta)\\ &\quad-\Upsilon(\alpha,\theta)\left[A(\alpha)+\sum_{i=1}^{N}B_i(\alpha)K_i(\alpha)\right]\\ &\quad-2\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha,\theta)\\ &\quad-\theta\left[Q(\alpha)+\sum_{i=1}^{N}K_i^T(\alpha)R_i(\alpha)K_i(\alpha)\right],\\ \frac{d}{d\alpha}\varrho(\alpha,\theta) &= -\varrho(\alpha,\theta)\,\mathrm{Tr}\left\{\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\right\},\end{aligned}$$

where the terminal conditions $\Upsilon(t_f,\theta)=\theta Q_f$ and $\varrho(t_f,\theta)=1$. Finally, the remaining backward-in-time differential equation satisfied by $\upsilon(\alpha,\theta)$ is given by

$$\frac{d}{d\alpha}\upsilon(\alpha,\theta) = -\mathrm{Tr}\left\{\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\right\},\ \upsilon(t_f,\theta)=0$$

which completes the proof.

Now it is ready to generate cost cumulants for the multi-person decision-making problem by looking at a MacLaurin series expansion of the cumulant-generating function

$$\begin{aligned}\psi(\alpha,x_\alpha;\theta) &= \sum_{j=1}^{\infty}\kappa_j(\alpha,x_\alpha)\frac{\theta^j}{j!}\\ &= \sum_{j=1}^{\infty}\frac{\partial^{(j)}}{\partial\theta^{(j)}}\psi(\alpha,x_\alpha;\theta)\bigg|_{\theta=0}\frac{\theta^j}{j!}\end{aligned}\qquad(14)$$

in which $\kappa_j(\alpha,x_\alpha)$'s are called the cost cumulants. Notice that the series coefficients can be computed by using (10)

$$\begin{aligned}&\frac{\partial^{(j)}}{\partial\theta^{(j)}}\psi(\alpha,x_\alpha;\theta)\bigg|_{\theta=0} =\\ &\quad x_\alpha^T\,\frac{\partial^{(j)}}{\partial\theta^{(j)}}\Upsilon(\alpha,\theta)\bigg|_{\theta=0}x_\alpha+\frac{\partial^{(j)}}{\partial\theta^{(j)}}\upsilon(\alpha,\theta)\bigg|_{\theta=0}.\end{aligned}\qquad(15)$$

In view of the results (14) and (15), cost cumulants for the stochastic decision problem can be obtained as

$$\kappa_j(\alpha,x_\alpha) = x_\alpha^T\frac{\partial^{(j)}}{\partial\theta^{(j)}}\Upsilon(\alpha,\theta)\bigg|_{\theta=0}x_\alpha+\frac{\partial^{(j)}}{\partial\theta^{(j)}}\upsilon(\alpha,\theta)\bigg|_{\theta=0}\qquad(16)$$

for any finite $1\le j<\infty$. For notational convenience, the following definitions are needed in place

$$H(\alpha,j)\triangleq\frac{\partial^{(j)}}{\partial\theta^{(j)}}\Upsilon(\alpha,\theta)\bigg|_{\theta=0};\ D(\alpha,j)\triangleq\frac{\partial^{(j)}}{\partial\theta^{(j)}}\upsilon(\alpha,\theta)\bigg|_{\theta=0}\qquad(17)$$

Theorem 2: Cost Cumulants in Decision Problems.
Let decision makers choose their control strategies $(u_1(t),\ldots,u_N(t))=(K_1(t)x(t),\ldots,K_N(t)x(t))$, where the dynamics of the multi-person single-objective decision system is governed by the linear stochastic differential equation (4) and is associated with finite-horizon IQF payoff functional (5). For $k\in\mathbb{Z}^+$ fixed and $1\le r\le k$, the kth-cost cumulant in the multi-person decision-making problem can be shown of the form

$$\kappa_k(t_0,x_0;K_1,\ldots,K_N) = x_0^T H(t_0,k)x_0 + D(t_0,k)\,,\qquad(18)$$

in which the cumulant-building variables $\{H(\alpha,r)\}_{r=1}^k$ and $\{D(\alpha,r)\}_{r=1}^k$ evaluated at $\alpha=t_0$ satisfy the following differential equations (with the dependence of $H(\alpha,r)$ and $D(\alpha,r)$ upon the admissible gains $K_1,\ldots,K_N$ suppressed)

$$\begin{aligned}\frac{d}{d\alpha}H(\alpha,1) &= -\left[A(\alpha)+\sum_{i=1}^{N}B_i(\alpha)K_i(\alpha)\right]^T H(\alpha,1)\\ &\quad-H(\alpha,1)\left[A(\alpha)+\sum_{i=1}^{N}B_i(\alpha)K_i(\alpha)\right]\\ &\quad-Q(\alpha)-\sum_{i=1}^{N}K_i^T(\alpha)R_i(\alpha)K_i(\alpha)\,,\end{aligned}\qquad(19)$$

and, for $2\le r\le k$

$$\begin{aligned}\frac{d}{d\alpha}H(\alpha,r) &= -\left[A(\alpha)+\sum_{i=1}^{N}B_i(\alpha)K_i(\alpha)\right]^T H(\alpha,r)\\ &\quad-H(\alpha,r)\left[A(\alpha)+\sum_{i=1}^{N}B_i(\alpha)K_i(\alpha)\right]\\ &-\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}H(\alpha,s)G(\alpha)WG^T(\alpha)H(\alpha,r-s)\,,\end{aligned}\qquad(20)$$

together with $1\le r\le k$

$$\frac{d}{d\alpha}D(\alpha,r) = -\mathrm{Tr}\left\{H(\alpha,r)G(\alpha)WG^T(\alpha)\right\},\qquad(21)$$

where the terminal conditions $H(t_f,1)=Q_f$, $H(t_f,r)=0$ for $2\le r\le k$ and $D(t_f,r)=0$ for $1\le r\le k$.

Proof. The cost cumulant expression in (18) is readily justified by using the result (16) and the definitions (17). What remains to show that the solutions $H(\alpha,r)$ and $D(\alpha,r)$ for $1\le r\le k$ indeed satisfy the equations (19)-(21). Note that the equations (19)-(21) are satisfied by the solutions $H(\alpha,r)$ and $D(\alpha,r)$ can be obtained by repeatedly taking the derivative with respect to θ of the equations (11)-(12) together with the assumption $A(\alpha)+\sum_{i=1}^N B_i(\alpha)K_i(\alpha)$, stable for all $\alpha\in[t_0,t_f]$.

III. Problem Statements

In the subsequent development, the subset of symmetric matrices of the vector space of all $n\times n$ matrices with real elements is denoted by $\mathbb{S}^n$. Now let k-tuple variables $\mathcal{H}$ and $\mathcal{D}$ be defined as follows $\mathcal{H}(\cdot)\triangleq(\mathcal{H}_1(\cdot),\ldots,\mathcal{H}_k(\cdot))$ and $\mathcal{D}(\cdot)\triangleq(\mathcal{D}_1(\cdot),\ldots,\mathcal{D}_k(\cdot))$ where each element $\mathcal{H}_r\in\mathcal{C}^1([t_0,t_f];\mathbb{S}^n)$ of $\mathcal{H}$ and $\mathcal{D}_r\in\mathcal{C}^1([t_0,t_f];\mathbb{R})$ of $\mathcal{D}$ have the representations $\mathcal{H}_r(\cdot)=H(\cdot,r)$ and $\mathcal{D}_r(\cdot)=D(\cdot,r)$ with the right members satisfying the dynamic equations (19)-(21) on the horizon $[t_0,t_f]$. For notational simplicity, the following convenient mappings become necessary

$$\begin{aligned}\mathcal{F}_r &: [t_0,t_f]\times(\mathbb{S}^n)^k\times\mathbb{R}^{m_1\times n}\times\cdots\times\mathbb{R}^{m_N\times n}\mapsto\mathbb{S}^n\\ \mathcal{G}_r &: [t_0,t_f]\times(\mathbb{S}^n)^k\mapsto\mathbb{R}\end{aligned}$$

with the actions given by

$$\mathcal{F}_1(\alpha,\mathcal{H},K_1,\ldots,K_2) \triangleq -\left[A(\alpha)+\sum_{i=1}^{N}B_i(\alpha)K_i(\alpha)\right]^T \mathcal{H}_1(\alpha) - \mathcal{H}_1(\alpha)\left[A(\alpha)+\sum_{i=1}^{N}B_i(\alpha)K_i(\alpha)\right] - Q(\alpha)-\sum_{i=1}^{N}K_i^T(\alpha)R_i(\alpha)K_i(\alpha),$$

and, for $2 \leq r \leq k$

$$\mathcal{F}_r(\alpha,\mathcal{H},K_1,\ldots,K_N) \triangleq -\left[A(\alpha)+\sum_{i=1}B_i(\alpha)K_i(\alpha)\right]^T \mathcal{H}_r(\alpha) - \mathcal{H}_r(\alpha)\left[A(\alpha)+\sum_{i=1}^{N}B_i(\alpha)K_i(\alpha)\right] - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_s(\alpha)G(\alpha)WG^T(\alpha)\mathcal{H}_{r-s}(\alpha),$$

finally, for $1 \leq r \leq k$

$$\mathcal{G}_r(\alpha,\mathcal{H}) \triangleq -\mathrm{Tr}\left\{\mathcal{H}_r(\alpha)G(\alpha)WG^T(\alpha)\right\}.$$

For a compact formulation, the product mappings should be established such that $\mathcal{F}_1 \times \cdots \times \mathcal{F}_k : [t_0,t_f]\times(\mathbb{S}^n)^k \times \mathbb{R}^{m_1\times n}\times\cdots\times\mathbb{R}^{m_N\times n} \mapsto (\mathbb{S}^n)^k$ and $\mathcal{G}_1\times\cdots\times\mathcal{G}_k : [t_0,t_f]\times(\mathbb{S}^n)^k \mapsto \mathbb{R}^k$ along with the corresponding notations $\mathcal{F}=\mathcal{F}_1\times\cdots\times\mathcal{F}_k$ and $\mathcal{G}=\mathcal{G}_1\times\cdots\times\mathcal{G}_k$. Thus, the dynamic equations of motion (19)-(21) can be rewritten as follows

$$\frac{d}{d\alpha}\mathcal{H}(\alpha)=\mathcal{F}(\alpha,\mathcal{H}(\alpha),K_1(\alpha),\ldots,K_N(\alpha)), \quad (22)$$

$$\frac{d}{d\alpha}\mathcal{D}(\alpha)=\mathcal{G}(\alpha,\mathcal{H}(\alpha)), \quad (23)$$

where the terminal values $\mathcal{H}(t_f) \triangleq \mathcal{H}_f = (Q_f,0,\ldots,0)$ and $\mathcal{D}(t_f) \triangleq \mathcal{D}_f = (0,\ldots,0)$.

Note that the product system uniquely determines $\mathcal{H}$ and $\mathcal{D}$ once the admissible feedback gains $K_1,\ldots,K_N$ are specified. Hence, $\mathcal{H}$ and $\mathcal{D}$ are considered as $\mathcal{H}(\cdot,K_1,\ldots,K_N)$ and $\mathcal{D}(\cdot,K_1,\ldots,K_N)$, respectively. The performance index in the cost-cumulant control problem can now be formulated in the admissible feedback gains $K_1,\ldots,K_N$.

Definition 1: Performance Index.
Fix $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_l \geq 0\}_{l=1}^k$ with $\mu_1 > 0$. Then for the given initial condition (t_0,x_0), the performance index $\phi_0 : [t_0,t_f]\times(\mathbb{S}^n)^k\times\mathbb{R}^k \mapsto \mathbb{R}^+$ of the cost-cumulant control is defined as

$$\phi_0\left(t_0,\mathcal{H}(t_0,K_1,\ldots,K_N),\mathcal{D}(t_0,K_1,\ldots,K_N)\right) \triangleq \sum_{l=1}^{k}\mu_l\kappa_l(K_1,\ldots,K_N) = \sum_{l=1}^{k}\mu_l\big[x_0^T\mathcal{H}_l(t_0,K_1,\ldots,K_N)x_0 + \mathcal{D}_l(t_0,K_1,\ldots,K_N)\big], \quad (24)$$

where real constants μ_l mutually chosen by cooperative decision makers represent different levels of influence as they deem important to the overall cost distribution and symmetric solutions $\{\mathcal{H}_l(t_0,K_1,\ldots,K_N) \geq 0\}_{l=1}^k$ and $\{\mathcal{D}_l(t_0,K_1,\ldots,K_N) \geq 0\}_{l=1}^k$ evaluated at $\alpha = t_0$ satisfy the equations (22)-(23).
For the given terminal data $(t_f,\mathcal{H}_f,\mathcal{D}_f)$, the classes $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu},\ldots,\mathcal{K}^N_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ of admissible feedback gains may be defined as follows.

Definition 2: Admissible Feedback Decision Strategies.
Let the compact subsets $\overline{K}_1 \subset \mathbb{R}^{m_1\times n},\ldots,\overline{K}_N \subset \mathbb{R}^{m_N\times n}$ be the sets of allowable gain values. For the given $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_l \geq 0\}_{l=1}^k$ with $\mu_1 > 0$, the sets of admissible decision strategies $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu},\ldots,\mathcal{K}^N_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ are assumed to be the classes of $\mathcal{C}([t_0,t_f];\mathbb{R}^{m_1\times n}),\ldots,\mathcal{C}([t_0,t_f];\mathbb{R}^{m_N\times n})$ with values $K_1(\cdot)\in\overline{K}_1,\ldots,K_N(\cdot)\in\overline{K}_N$ for which solutions to the dynamic equations of motion (22)-(23) exist on the finite horizon $[t_0,t_f]$.

Definition 3: Optimization Problem.
Suppose that $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_l \geq 0\}_{l=1}^k$ with $\mu_1 > 0$ are fixed. Then the cost-cumulant control optimization problem over $[t_0,t_f]$ is given by the minimization of the performance index (24) for all $K_1(\cdot)\in\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu},\ldots,K_N(\cdot)\in\mathcal{K}^N_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ and subject to the dynamic equations (22)-(23) for $\alpha \in [t_0,t_f]$.
Let's now introduce the value function, $\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})$ for the decision problem starting at the time-states triple $(\varepsilon,\mathcal{Y},\mathcal{Z})$.

Definition 4: Value Function.
The value function $\mathcal{V} : [t_0,t_f]\times(\mathbb{S}^n)^k\times\mathbb{R}^k \mapsto \mathbb{R}^+\cup\{+\infty\}$ associated with the Mayer problem is defined by

$$\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) \triangleq \min_{K_1(\cdot)\in\mathcal{K}^1_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu},\ldots,K_N(\cdot)\in\mathcal{K}^2_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}}\phi_0(\cdot,\cdot,\cdot),$$

for any $(\varepsilon,\mathcal{Y},\mathcal{Z}) \in [t_0,t_f]\times(\mathbb{S}^n)^k\times\mathbb{R}^k$.
Conventionally, set $\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) = \infty$ when any of $\mathcal{K}^1_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu},\ldots,\mathcal{K}^N_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}$ is empty. The development in the sequel is motivated by the excellent treatment in [2], and is intended to follow it closely. Unless otherwise specified, the dependence of trajectory solutions $\mathcal{H}$ and $\mathcal{D}$ on the admissible gains $K_1,\ldots,K_N$ is now omitted for notational clarity.

Theorem 3: Property 1: Necessary Condition.
The value function evaluated along any trajectory corresponding to a pair of control strategy gains feasible for its terminal states is a non-increasing function of time.

Theorem 4: Property 2: Necessary Condition.
The value function evaluated along any optimal trajectory is constant.
It is important to note that these properties are necessary conditions for optimality. The next theorem shows that these conditions are also sufficient for optimality.

Theorem 5: Sufficient Condition.
Let $\mathcal{W}(\varepsilon,\mathcal{Y},\mathcal{Z})$ be an extended real-valued function defined on $[t_0,t_f]\times(\mathbb{S}^n)^k\times\mathbb{R}^k$ such that $\mathcal{W}(\varepsilon,\mathcal{Y},\mathcal{Z}) = \phi_0(\varepsilon,\mathcal{Y},\mathcal{Z})$.

Let t_f, $\mathcal{H}_f$, $\mathcal{D}_f$ be given terminal conditions and let, for each trajectory pair $(\mathcal{H},\mathcal{D})$ corresponding to the decision strategies $(K_1,\dots,K_N)$ in $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}\times\cdots\times\mathcal{K}^N_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$, $\mathcal{W}(\alpha,\mathcal{H}(\alpha),\mathcal{D}(\alpha))$ be finite and non-increasing on $[t_0,t_f]$.

If $(K_1^*,\dots,K_N^*)$ are decision strategies in $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}\times\cdots\times\mathcal{K}^N_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ such that for the corresponding trajectory pair $(\mathcal{H}^*,\mathcal{D}^*)$, $\mathcal{W}(\alpha,\mathcal{H}^*(\alpha),\mathcal{D}^*(\alpha))$ is constant then $(K_1^*,\dots,K_N^*)$ are optimal strategies and $\mathcal{W}(t_f,\mathcal{H}_f,\mathcal{D}_f)=\mathcal{V}(t_f,\mathcal{H}_f,\mathcal{D}_f)$.

Corollary 1: Restriction of Decision Strategies.
Let $(K_1^*,\dots,K_N^*)$ be optimal decision strategies in $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}\times\cdots\times\mathcal{K}^N_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ and $(\mathcal{H}^*,\mathcal{D}^*)$ the corresponding trajectory pair of dynamic equations

$$\frac{d}{d\alpha}\mathcal{H}(\alpha)=\mathcal{F}(\alpha,\mathcal{H}(\alpha),K_1(\alpha),\dots,K_N(\alpha))\,,\quad \mathcal{H}(t_f)$$
$$\frac{d}{d\alpha}\mathcal{D}(\alpha)=\mathcal{G}(\alpha,\mathcal{H}(\alpha))\,,\quad \mathcal{D}(t_f)\,.$$

Then the restriction of $(K_1^*,\dots,K_N^*)$ to $[t_0,\alpha]$ are optimal decision strategies for each control problem with terminal conditions $(\alpha,\mathcal{H}^*(\alpha),\mathcal{D}^*(\alpha))$ when $t_0\le\alpha\le t_f$.

Remarks. Both necessary and sufficient conditions implied by these properties for a control gain to be optimal give hints that one may find a function $\mathcal{W}(\varepsilon,\mathcal{Y},\mathcal{Z}):[t_0,t_f]\times(\mathbb{S}^n)^k\times\mathbb{R}^k\mapsto\mathbb{R}^+$ such that $\mathcal{W}(\varepsilon,\mathcal{Y},\mathcal{Z})=\phi_0(\varepsilon,\mathcal{Y},\mathcal{Z})$, $\mathcal{W}(\varepsilon,\mathcal{Y},\mathcal{Z})$ is constant on the corresponding trajectory pair, and $\mathcal{W}(\varepsilon,\mathcal{Y},\mathcal{Z})$ is non-increasing on other trajectories.

Note that the value function $\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})$ is supposed to be continuously differentiable in $(\varepsilon,\mathcal{Y},\mathcal{Z})$. Formally speaking, the result regarding the differentiability of the value function adapted from [2] is stated as follows.

Theorem 6: Differentiability of Value Function.
Let $K_1^*(\alpha,\mathcal{H},\mathcal{D})$, $K_2^*(\alpha,\mathcal{H},\mathcal{D})$ $,\dots,$ $K_N^*(\alpha,\mathcal{H},\mathcal{D})$, $t_0(\varepsilon,\mathcal{Y},\mathcal{Z})$, and $(\mathcal{H}(t_0(\varepsilon,\mathcal{Y},\mathcal{Z});\varepsilon,\mathcal{Y}),\mathcal{D}(t_0(\varepsilon,\mathcal{Y},\mathcal{Z});\varepsilon,\mathcal{Z}))$ be optimal decision laws, an initial time and initial states for the trajectories of

$$\frac{d}{d\alpha}\mathcal{H}(\alpha)=\mathcal{F}(\alpha,\mathcal{H},K_1^*(\alpha,\mathcal{H},\mathcal{D}),\dots,K_N^*(\alpha,\mathcal{H},\mathcal{D}))\,,$$
$$\frac{d}{d\alpha}\mathcal{D}(\alpha)=\mathcal{G}(\alpha,\mathcal{H})\,,$$

with the terminal condition $(\varepsilon,\mathcal{Y},\mathcal{Z})$. Then, the value function $\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})$ is differentiable at each point at which $t_0(\varepsilon,\mathcal{Y},\mathcal{Z})$ and $\mathcal{H}(t_0(\varepsilon,\mathcal{Y},\mathcal{Z});\varepsilon,\mathcal{Y})$ and $\mathcal{D}(t_0(\varepsilon,\mathcal{Y},\mathcal{Z});\varepsilon,\mathcal{Z})$ are differentiable with respect to $(\varepsilon,\mathcal{Y},\mathcal{Z})$.

Definition 5: Playable Set.
Let the playable set $\mathcal{Q}$ be defined as follows

$$\mathcal{Q}\triangleq\Big\{(\varepsilon,\mathcal{Y},\mathcal{Z})\in[t_0,t_f]\times(\mathbb{S}^n)^k\times\mathbb{R}^k \\ \text{such that}\,\mathcal{K}^1_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}\times\cdots\times\mathcal{K}^N_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}\neq 0\Big\}.$$

Theorem 7: HJB Equation-Mayer Problem.
Let $(\varepsilon,\mathcal{Y},\mathcal{Z})$ be any interior point of the playable set $\mathcal{Q}$ at which the value function $\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})$ is differentiable. Then $\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})$ satisfies the partial differential inequality

$$0\ge\frac{\partial}{\partial\varepsilon}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) +\frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Y})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})\cdot\mathrm{vec}(\mathcal{F}(\varepsilon,\mathcal{Y},K_1,\dots,K_N)) +\frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Z})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})\cdot\mathrm{vec}(\mathcal{G}(\varepsilon,\mathcal{Y}))\,,$$

for all $(K_1,\dots,K_N)\in\overline{K}_1\times\cdots\times\overline{K}_N$.

If there exist optimal decision strategies $(K_1^*,\dots,K_N^*)\in\mathcal{K}^1_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}\times\cdots\times\mathcal{K}^N_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}$, then the partial differential equation of decision problems

$$0=\min_{K_1\in\overline{K}_1,\dots,K_N\in\overline{K}_N}\Big\{\frac{\partial}{\partial\varepsilon}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) +\frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Y})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})\cdot\mathrm{vec}(\mathcal{F}(\varepsilon,\mathcal{Y},K_1,\dots,K_N)) +\frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Z})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})\cdot\mathrm{vec}(\mathcal{G}(\varepsilon,\mathcal{Y}))\Big\} \quad (25)$$

is satisfied together with $\mathcal{V}(t_0,\mathcal{H}_0,\mathcal{D}_0)=\phi_0(t_0,\mathcal{H}_0,\mathcal{D}_0)$ and $\mathrm{vec}(\cdot)$ the vectorizing operator of enclosed entities. The optimum in (25) is achieved by the left limit $(K_1^*(\varepsilon)^-,\dots,K_N^*(\varepsilon)^-)$ of the optimal strategies at ε.

Theorem 8: Verification Theorem.
Fix $k\in\mathbb{Z}^+$. Let $\mathcal{W}(\varepsilon,\mathcal{Y},\mathcal{Z})$ be a continuously differentiable solution of the HJB equation

$$0=\min_{K_1\in\overline{K}_1,\dots,K_N\in\overline{K}_N}\Big\{\frac{\partial}{\partial\varepsilon}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) +\frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Y})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})\cdot\mathrm{vec}(\mathcal{F}(\varepsilon,\mathcal{Y},K_1,\dots,K_N)) +\frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Z})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})\cdot\mathrm{vec}(\mathcal{G}(\varepsilon,\mathcal{Y}))\Big\}$$

and satisfy the boundary condition for all $(t_0,\mathcal{H}_0,\mathcal{D}_0)\in\mathcal{M}$

$$\mathcal{W}(t_0,\mathcal{H}_0,\mathcal{D}_0)=\phi_0\left(t_0,\mathcal{H}_0,\mathcal{D}_0\right)\,, \quad (26)$$

where $\mathcal{M}\triangleq\{t_0\}\times(\mathbb{S}^n)^k\times\mathbb{R}^k$.

Let $(t_f,\mathcal{H}_f,\mathcal{D}_f)$ be a point of $\mathcal{Q}$, $(K_1,\dots,K_N)$ control strategies in $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}\times\cdots\times\mathcal{K}^N_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ and $\mathcal{H}$ and $\mathcal{D}$ the corresponding solutions of the equations

$$\frac{d}{d\alpha}\mathcal{H}(\alpha)=\mathcal{F}(\alpha,\mathcal{H}(\alpha),K_1(\alpha),\dots,K_N(\alpha))\,,\quad \mathcal{H}(t_f)$$
$$\frac{d}{d\alpha}\mathcal{D}(\alpha)=\mathcal{G}(\alpha,\mathcal{H}(\alpha))\,,\quad \mathcal{D}(t_f)\,.$$

Then $\mathcal{W}(\alpha,\mathcal{H}(\alpha),\mathcal{D}(\alpha))$ is a non-increasing function of α. If $(K_1^*,\dots,K_N^*)$ are control strategies in $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}\times\cdots\times\mathcal{K}^N_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ defined on $[t_0,t_f]$ with corresponding solution,

$\mathcal{H}^*$ and $\mathcal{D}^*$ of the above equations such that for $\alpha \in [t_0, t_f]$

$$0 = \frac{\partial}{\partial \varepsilon}\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha)) + \frac{\partial}{\partial\, \mathrm{vec}(\mathcal{Y})}\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha)) \cdot \mathrm{vec}(\mathcal{F}(\alpha, \mathcal{H}^*(\alpha), K_1^*(\alpha), \ldots, K_N^*(\alpha))) + \frac{\partial}{\partial\, \mathrm{vec}(\mathcal{Z})}\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha))\mathrm{vec}(\mathcal{G}(\alpha, \mathcal{H}^*(\alpha))) \quad (27)$$

then $(K_1^*, \ldots, K_N^*)$ are optimal decision strategies in $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu} \times \cdots \times \mathcal{K}^N_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ and

$$\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = \mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})\,, \quad (28)$$

where $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is the value function.

It is observed that to have the an optimal solution along with the decision laws $(K_1^*, \ldots, K_N^*) \in \mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu} \times \cdots \times \mathcal{K}^N_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ well defined and continuous for all $\alpha \in [t_0, t_f]$, the solution $\mathcal{H}(\alpha)$ to the equation (22) when evaluated at $\alpha = t_0$ must then exist. Therefore, it is necessary that $\mathcal{H}(\alpha)$ is finite for all $\alpha \in [t_0, t_f)$. Moreover, the solution of the equation (22) exists and is continuously differentiable in a neighborhood of t_f. Applying the results from [1], these solutions can further be extended to the left of t_f as long as $\mathcal{H}(\alpha)$ remains finite. Hence, the existences of unique and continuously differentiable solutions to the equation (22) are certain if $\mathcal{H}(\alpha)$ are bounded for all $\alpha \in [t_0, t_f)$. As the result, the candidate value functions $\mathcal{V}(\alpha, \mathcal{H}, \mathcal{D})$ are continuously differentiable as well. The following theorem is proven.

Theorem 9: Necessary and Sufficient Conditions.
$(K_1^*, \ldots, K_N^*)$ are optimal strategies if and only if $\mathcal{H}(\alpha)$ is bounded for all $\alpha \in [t_0, t_f)$.

IV. Strategies by Cooperative Decision Makers

Recall that the optimization problem being considered herein is in "Mayer form" and can be solved by applying an adaptation of the Mayer form verification theorem of dynamic programming given in [2]. In the framework of dynamic programming, it is often required to denote the terminal time and states of a family of optimization problems as $(\varepsilon, \mathcal{Y}, \mathcal{Z})$ rather than $(t_f, \mathcal{H}_f, \mathcal{D}_f)$. That is, for $\varepsilon \in [t_0, t_f]$ and $1 \le l \le k$, the states of the system (22)-(23) defined on the interval $[t_0, \varepsilon]$ have the terminal values denoted by $\mathcal{H}(\varepsilon) \equiv \mathcal{Y}$ and $\mathcal{D}(\varepsilon) \equiv \mathcal{Z}$. Since the performance index (24) is quadratic affine in terms of arbitrarily fixed x_0, this observation suggests a solution to the HJB equation (25) may be of the form as follows.

Theorem 10: Candidate Value-Function.
Fix $k \in \mathbb{Z}^+$ and let $(\varepsilon, \mathcal{Y}, \mathcal{Z})$ be any interior point of the reachable set $\mathcal{Q}$ at which the real-valued function

$$\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = x_0^T \sum_{l=1}^{k} \mu_l(\mathcal{Y}_l + \mathcal{E}_l(\varepsilon))\, x_0 + \sum_{l=1}^{k} \mu_l(\mathcal{Z}_l + \mathcal{T}_l(\varepsilon)) \quad (29)$$

is differentiable. The parametric functions of time $\mathcal{E}_l \in \mathcal{C}^1([t_0, t_f]; \mathbb{S}^n)$ and $\mathcal{T}_l \in \mathcal{C}^1([t_0, t_f]; \mathbb{R})$ are yet to be determined. Furthermore, the time derivative of $\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is

$$\frac{d}{d\varepsilon}\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = \sum_{l=1}^{k} \mu_l \left(\mathcal{G}_l(\varepsilon, \mathcal{Y}) + \frac{d}{d\varepsilon}\mathcal{T}_l(\varepsilon)\right) + x_0^T \sum_{l=1}^{k} \mu_l \left(\mathcal{F}_l(\varepsilon, \mathcal{Y}, K_1, \ldots, K_N) + \frac{d}{d\varepsilon}\mathcal{E}_l(\varepsilon)\right) x_0. \quad (30)$$

The substitution of this hypothesized solution (29) into the HJB equation (25) and making use of the result (30) yields

$$\begin{aligned} 0 &= \min_{K_1 \in \overline{K}_1, \ldots, K_N \in \overline{K}_N} \left\{ \frac{\partial}{\partial \varepsilon}\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) + \frac{\partial}{\partial\, \mathrm{vec}(\mathcal{Y})}\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \cdot \mathrm{vec}(\mathcal{F}(\varepsilon, \mathcal{Y}, K_1, \ldots, K_N)) + \frac{\partial}{\partial\, \mathrm{vec}(\mathcal{Z})}\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \cdot \mathrm{vec}(\mathcal{G}(\varepsilon, \mathcal{Y})) \right\}, \\ &= \min_{K_1 \in \overline{K}_1, \ldots, K_N \in \overline{K}_N} \left\{ x_0^T \left(\sum_{l=1}^{k} \mu_l \frac{d}{d\varepsilon}\mathcal{E}_l(\varepsilon)\right) x_0 + \sum_{l=1}^{k} \mu_l \frac{d}{d\varepsilon}\mathcal{T}_l(\varepsilon) + \sum_{l=1}^{k} \mu_l \mathcal{G}_l(\varepsilon, \mathcal{Y}) + x_0^T \left(\sum_{l=1}^{k} \mu_l \mathcal{F}_l(\varepsilon, \mathcal{Y}, K_1, \ldots, K_N)\right) x_0 \right\}. \end{aligned} \quad (31)$$

Differentiating the expression within the bracket of (31) with respect to $K_1, \ldots, K_N$ yield the necessary conditions for an extremum of the performance index (24) on $[t_0, \varepsilon]$,

$$\begin{aligned} -2B_1^T(\varepsilon) \sum_{l=1}^{k} \mu_l \mathcal{Y}_l M_0 - 2\mu_1 R_1(\varepsilon) K_1 M_0 &= 0\,, \\ \vdots &= \vdots \\ -2B_N^T(\varepsilon) \sum_{l=1}^{k} \mu_l \mathcal{Y}_l M_0 - 2\mu_1 R_N(\varepsilon) K_N M_0 &= 0\,. \end{aligned}$$

Because M_0 is an arbitrary rank-one matrix, it must be true

$$K_1(\varepsilon, \mathcal{Y}, \mathcal{Z}) = -R_1^{-1}(\varepsilon) B_1^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{Y}_r\,, \quad (32)$$

$$\vdots = \vdots$$

$$K_N(\varepsilon, \mathcal{Y}, \mathcal{Z}) = -R_N^{-1}(\varepsilon) B_N^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{Y}_r\,, \quad (33)$$

where $\widehat{\mu}_r \triangleq \mu_l/\mu_1$ for $\mu_1 > 0$. Substituting the gain expressions (32) and (33) into the right member of the HJB equation (31) yields the value of the minimum

$$x_0^T \left[\sum_{l=1}^{k} \mu_l \frac{d}{d\varepsilon}\mathcal{E}_l(\varepsilon) - A^T(\varepsilon) \sum_{l=1}^{k} \mu_l \mathcal{Y}_l - \sum_{l=1}^{k} \mu_l \mathcal{Y}_l A(\varepsilon) - \mu_1 Q(\varepsilon) + \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{Y}_r \left[\sum_{i=1}^{N} B_i(\varepsilon) R_i^{-1}(\varepsilon) B_i^T(\varepsilon)\right] \sum_{l=1}^{k} \mu_l \mathcal{Y}_l \right.$$

$$
\begin{aligned}
&+\sum_{l=1}^{k}\mu_l\mathcal{Y}_l\left[\sum_{i=1}^{N}B_i(\varepsilon)R_i^{-1}(\varepsilon)B_i^T(\varepsilon)\right]\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{Y}_s\\
&-\mu_1\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{Y}_r\left[\sum_{i=1}^{N}B_i(\varepsilon)R_i^{-1}(\varepsilon)B_i^T(\varepsilon)\right]\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{Y}_s\\
&-\sum_{l=2}^{k}\mu_l\sum_{q=1}^{l-1}\frac{2l!}{q!(l-q)!}\mathcal{Y}_qG(\varepsilon)WG^T(\varepsilon)\mathcal{Y}_{l-q}\Bigg]x_0\\
&+\sum_{l=1}^{k}\mu_l\frac{d}{d\varepsilon}\mathcal{T}_l(\varepsilon)-\sum_{l=1}^{k}\mu_l\mathrm{Tr}\left\{\mathcal{Y}_lG(\varepsilon)WG^T(\varepsilon)\right\}\,. \quad (34)
\end{aligned}
$$

It is now necessary to exhibit $\{\mathcal{E}_p(\cdot)\}_{p=1}^k$ and $\{\mathcal{T}_p(\cdot)\}_{p=1}^k$ which render the left side of (34) equal to zero for $\varepsilon \in [t_0, t_f]$, when $\{\mathcal{Y}_p\}_{p=1}^k$ are evaluated along solution trajectories. Studying the expression (34) reveals that $\mathcal{E}_p(\cdot)$ and $\mathcal{T}_p(\cdot)$ for $1 \le p \le k$ satisfying the differential equations

$$
\begin{aligned}
\frac{d}{d\varepsilon}\mathcal{E}_1(\varepsilon) &= A^T(\varepsilon)\mathcal{H}_1(\varepsilon)+\mathcal{H}_1(\varepsilon)A(\varepsilon)+Q(\varepsilon)\\
&-\mathcal{H}_1(\varepsilon)\left[\sum_{i=1}^{N}B_i(\varepsilon)R_i^{-1}(\varepsilon)B_i^T(\varepsilon)\right]\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)\\
&-\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)\left[\sum_{i=1}^{N}B_i(\varepsilon)R_i^{-1}(\varepsilon)B_i^T(\varepsilon)\right]\mathcal{H}_1(\varepsilon)\\
&+\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)\left[\sum_{i=1}^{N}B_i(\varepsilon)R_i^{-1}(\varepsilon)B_i^T(\varepsilon)\right]\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) \quad (35)
\end{aligned}
$$

and, for $2 \le p \le k$

$$
\begin{aligned}
\frac{d}{d\varepsilon}\mathcal{E}_p(\varepsilon) &= A^T(\varepsilon)\mathcal{H}_p(\varepsilon)+\mathcal{H}_p(\varepsilon)A(\varepsilon)\\
&-\mathcal{H}_p(\varepsilon)\left[\sum_{i=1}^{N}B_i(\varepsilon)R_i^{-1}(\varepsilon)B_i^T(\varepsilon)\right]\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)\\
&-\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)\left[\sum_{i=1}^{N}B_i(\varepsilon)R_i^{-1}(\varepsilon)B_i^T(\varepsilon)\right]\mathcal{H}_p(\varepsilon)\\
&+\sum_{q=1}^{p-1}\frac{2p!}{q!(p-q)!}\mathcal{H}_q(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\mathcal{H}_{p-q}(\varepsilon)\,, \quad (36)
\end{aligned}
$$

together with, for $1 \le p \le k$

$$
\frac{d}{d\varepsilon}\mathcal{T}_p(\varepsilon) = \mathrm{Tr}\left\{\mathcal{H}_p(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\right\} \quad (37)
$$

will work. Furthermore, at the boundary condition, it is necessary to have $\mathcal{W}(t_0,\mathcal{H}_0,\mathcal{D}_0) = \phi_0(t_0,\mathcal{H}_0,\mathcal{D}_0)$. Or, equivalently

$$
\begin{aligned}
x_0^T\sum_{l=1}^{k}\mu_l(\mathcal{H}_{l0}+\mathcal{E}_l(t_0))\,x_0+\sum_{l=1}^{k}\mu_l(\mathcal{D}_{l0}+\mathcal{T}_l(t_0)) &=\\
x_0^T\sum_{l=1}^{k}\mu_l\mathcal{H}_{l0}x_0+\sum_{l=1}^{k}\mu_l\mathcal{D}_{l0}\,.&
\end{aligned}
$$

Thus, matching the boundary condition yields the corresponding initial value conditions $\mathcal{E}_p(t_0)=0$ and $\mathcal{T}_p(t_0)=0$ for the equations (35)-(37). Applying the feedback gain specified in (32) and (33) along the solution trajectories of the equations (22)-(23), these equations become

$$
\begin{aligned}
\frac{d}{d\varepsilon}\mathcal{H}_1(\varepsilon) &= -A^T(\varepsilon)\mathcal{H}_1(\varepsilon)-\mathcal{H}_1(\varepsilon)A(\varepsilon)-Q(\varepsilon)\\
&+\mathcal{H}_1(\varepsilon)\left[\sum_{i=1}^{N}B_i(\varepsilon)R_i^{-1}(\varepsilon)B_i^T(\varepsilon)\right]\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)\\
&+\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)\left[\sum_{i=1}^{N}B_i(\varepsilon)R_i^{-1}(\varepsilon)B_i^T(\varepsilon)\right]\mathcal{H}_1(\varepsilon)\\
&-\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)\left[\sum_{i=1}^{N}B_i(\varepsilon)R_i^{-1}(\varepsilon)B_i^T(\varepsilon)\right]\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) \quad (38)
\end{aligned}
$$

and, for $2 \le p \le k$

$$
\begin{aligned}
\frac{d}{d\varepsilon}\mathcal{H}_p(\varepsilon) &= -A^T(\varepsilon)\mathcal{H}_p(\varepsilon)-\mathcal{H}_p(\varepsilon)A(\varepsilon)\\
&+\mathcal{H}_p(\varepsilon)\left[\sum_{i=1}^{N}B_i(\varepsilon)R_i^{-1}(\varepsilon)B_i^T(\varepsilon)\right]\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)\\
&+\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)\left[\sum_{i=1}^{N}B_i(\varepsilon)R_i^{-1}(\varepsilon)B_i^T(\varepsilon)\right]\mathcal{H}_p(\varepsilon)\\
&-\sum_{q=1}^{p-1}\frac{2p!}{q!(p-q)!}\mathcal{H}_q(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\mathcal{H}_{p-q}(\varepsilon)\,, \quad (39)
\end{aligned}
$$

together with, for $1 \le p \le k$

$$
\frac{d}{d\varepsilon}\mathcal{D}_p(\varepsilon) = -\mathrm{Tr}\left\{\mathcal{H}_p(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\right\} \quad (40)
$$

where the terminal conditions $\mathcal{H}_1(t_f)=Q_f$, $\mathcal{H}_p(t_f)=0$ for $2 \le p \le k$ and $\mathcal{D}_p(t_f)=0$ for $1 \le p \le k$. Thus, whenever these equations (38)-(40) admit solutions $\{\mathcal{H}_p(\cdot)\}_{p=1}^k$ and $\{\mathcal{D}_p(\cdot)\}_{p=1}^k$, then the existence of $\{\mathcal{E}_p(\cdot)\}_{p=1}^k$ and $\{\mathcal{T}_p(\cdot)\}_{p=1}^k$ satisfying the equations (35)-(37) are assured. By comparing equations (35)-(37) to those of (38)-(40), one may recognize that these sets of equations are related to one another by $\frac{d}{d\varepsilon}\mathcal{E}_p(\varepsilon) = -\frac{d}{d\varepsilon}\mathcal{H}_p(\varepsilon)$ and $\frac{d}{d\varepsilon}\mathcal{T}_p(\varepsilon) = -\frac{d}{d\varepsilon}\mathcal{D}_p(\varepsilon)$ for $1 \le p \le k$. Enforcing the initial value conditions of $\mathcal{E}_p(t_0)=0$ and $\mathcal{T}_p(t_0)=0$ uniquely implies that $\mathcal{E}_p(\varepsilon)=\mathcal{H}_p(t_0)-\mathcal{H}_p(\varepsilon)$ and $\mathcal{T}_p(\varepsilon)=\mathcal{D}_p(t_0)-\mathcal{D}_p(\varepsilon)$ for all $\varepsilon \in [t_0,t_f]$ and yields a value function

$$
\begin{aligned}
\mathcal{W}(\varepsilon,\mathcal{Y},\mathcal{Z}) &= \mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})\\
&= x_0^T\sum_{l=1}^{k}\mu_l\mathcal{H}_l(t_0)x_0+\sum_{l=1}^{k}\mu_l\mathcal{D}_l(t_0)\,,
\end{aligned}
$$

for which the sufficient condition (27) of the verification theorem is satisfied. Therefore, the optimal decision laws for the decision maker 1, (32) and the decision maker N, (33) minimizing the performance index stated in (24) become

$$
K_1^*(\varepsilon) = -R_1^{-1}(\varepsilon)B_1^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r^*(\varepsilon)\,, \quad (41)
$$

$$
\vdots = \vdots
$$

$$
K_N^*(\varepsilon) = -R_N^{-1}(\varepsilon)B_N^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r^*(\varepsilon)\,. \quad (42)
$$

Theorem 11: Multi-Cumulant Cooperative Strategies. Consider the multi-person linear-quadratic differential system (4)-(5) whose pairs $(A, B_i), \dots, (A, B_N)$ are uniformly stabilizable on $[t_0, t_f]$. Let $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$. Then the optimal decision laws are achieved by the cooperative feedback control gains

$$K_1^*(\alpha) = -R_1^{-1}(\alpha) B_1^T(\alpha) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r^*(\alpha) , \tag{43}$$

$$\vdots = \vdots$$

$$K_N^*(\alpha) = -R_N^{-1}(\alpha) B_N^T(\alpha) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r^*(\alpha) , \tag{44}$$

where $\widehat{\mu}_r \triangleq \mu_l/\mu_1$ mutually chosen by cooperative decision makers represent different levels of influence as they deem important to the overall cost distribution and $\{\mathcal{H}_r^*(\alpha) \geq 0\}_{r=1}^k$ are the optimal solutions of the backward-in-time coupled Riccati-type differential equations

$$\begin{aligned} \frac{d}{d\alpha}\mathcal{H}_1^*(\alpha) = &-\left[A(\alpha) + \sum_{i=1}^{N} B_i(\alpha) K_i^*(\alpha)\right]^T \mathcal{H}_1^*(\alpha) \\ &- \mathcal{H}_1^*(\alpha)\left[A(\alpha) + \sum_{i=1}^{N} B_i(\alpha) K_i^*(\alpha)\right] \\ &- Q(\alpha) - \sum_{i=1}^{N} K_i^{*T}(\alpha) R_i(\alpha) K_i^*(\alpha) , \end{aligned} \tag{45}$$

and, for $2 \leq r \leq k$

$$\begin{aligned} \frac{d}{d\alpha}\mathcal{H}_r^*(\alpha) = &-\left[A(\alpha) + \sum_{i=1}^{N} B_i(\alpha) K_i^*(\alpha)\right]^T \mathcal{H}_r^*(\alpha) \\ &- \mathcal{H}_r^*(\alpha)\left[A(\alpha) + \sum_{i=1}^{N} B_i(\alpha) K_i^*(\alpha)\right] \\ &- \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!} \mathcal{H}_s^*(\alpha) G(\alpha) W G^T(\alpha) \mathcal{H}_{r-s}^*(\alpha) \end{aligned} \tag{46}$$

with the terminal boundary conditions $\mathcal{H}_1^*(t_f) = Q_f$, and $\mathcal{H}_r^*(t_f) = 0$ when $2 \leq r \leq k$.

In a situation where cooperative decision makers not only minimize the overall performance index of a total system but also, at the same time ensure that the closed-loop poles lie to the left of a line $Re(j\omega) = -\sigma$, for a prescribed $\sigma \in \mathbb{R}^+$. The advantages offered by this additional consideration include system robustness against variations of system parameters as well as tolerances of time delay and nonlinearities in the closed loop.

In place of the original cost (1), the new cost with a prescribed degree of stability $\sigma > 0$ is given by

$$\begin{aligned} J(t_0, x_0; u_1, \dots, u_N) &= x^T(t_f) Q_f e^{2\sigma t_f} x(t_f) \\ &+ \int_{t_0}^{t_f} \left[x^T(\tau) Q(\tau) x(\tau) + \sum_{i=1}^{N} u_i^T(\tau) R_i(\tau) u_i(\tau)\right] e^{2\sigma\tau} d\tau . \end{aligned} \tag{47}$$

Intuitively, the new control optimization can be converted to the original optimization problem with some changes of variables: $x_\sigma(t) \triangleq x(t)e^{\sigma t}$, $u_\sigma(t) \triangleq u(t)e^{\sigma t}$, and $w_\sigma(t) \triangleq w(t)e^{\sigma t}$. The strategy solutions are summarized as follows.

Theorem 12: Strategies with a Prescribed Stability. Consider the multi-person linear-quadratic differential system (4)-(5) whose pairs (A, B_i) are uniformly stabilizable on $[t_0, t_f]$. Let $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$. Then the optimal decision laws with a prescribed degree of stability, $\sigma > 0$ are achieved by cooperative gains

$$K_{\sigma,1}^*(\alpha) = -R_1^{-1}(\alpha) B_1^T(\alpha) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{\sigma,r}^*(\alpha) , \tag{48}$$

$$\vdots = \vdots$$

$$K_{\sigma,N}^*(\alpha) = -R_N^{-1}(\alpha) B_N^T(\alpha) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{\sigma,r}^*(\alpha) , \tag{49}$$

where $\widehat{\mu}_r \triangleq \mu_l/\mu_1$ represent different levels of influence as they deem important to the overall cost distribution and $\{\mathcal{H}_{\sigma,r}^*(\alpha) \geq 0\}_{r=1}^k$ are the optimal solutions of the equations

$$\begin{aligned} \frac{d}{d\alpha}\mathcal{H}_{\sigma,1}^*(\alpha) = &-\left[A(\alpha) + \sigma I + \sum_{i=1}^{N} B_i(\alpha) K_{\sigma,i}^*(\alpha)\right]^T \mathcal{H}_{\sigma,1}^*(\alpha) \\ &- \mathcal{H}_{\sigma,1}^*(\alpha)\left[A(\alpha) + \sigma I + \sum_{i=1}^{N} B_i(\alpha) K_{\sigma,i}^*(\alpha)\right] \\ &- Q(\alpha) - \sum_{i=1}^{N} K_{\sigma,i}^{*T}(\alpha) R_i(\alpha) K_{\sigma,i}^*(\alpha) , \end{aligned} \tag{50}$$

and, for $2 \leq r \leq k$

$$\begin{aligned} \frac{d}{d\alpha}\mathcal{H}_{\sigma,r}^*(\alpha) = &-\left[A(\alpha) + \sigma I + \sum_{i=1}^{N} B_i(\alpha) K_{\sigma,i}^*(\alpha)\right]^T \mathcal{H}_{\sigma,r}^*(\alpha) \\ &- \mathcal{H}_{\sigma,r}^*(\alpha)\left[A(\alpha) + \sigma I + \sum_{i=1}^{N} B_i(\alpha) K_{\sigma,i}^*(\alpha)\right] \\ &- \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!} \mathcal{H}_{\sigma,s}^*(\alpha) G(\alpha) W_\sigma G^T(\alpha) \mathcal{H}_{\sigma,r-s}^*(\alpha) \end{aligned} \tag{51}$$

with the terminal boundary conditions $\mathcal{H}_{\sigma,1}^*(t_f) = Q_f$, and $\mathcal{H}_{\sigma,r}^*(t_f) = 0$ when $2 \leq r \leq k$.

V. Conclusions

The recent results offer nontrivial educational and pedagogical contributions as well as performance analysis tools toward the establishment of new, statistically based design procedures for cooperative decision problems and stochastic games. Their practicality may be found in network-enabled collaborative systems, multi-layered sensing, and single integrated situational awareness applications.

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